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Shaghayegh. Khodkameh¹,
Mohammad Hassan. Behzadi^{1*},
Mohsen. Rostamy Malkhalifeh²

1 Department of Statistics, SR.C., Islamic Azad University, Tehran, Iran.

2 Department of mathematics and Computer Science, SR.C, Islamic Azad University, Tehran, Iran.

Corresponding author email address:
behzadi.mh@iau.ac.ir

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Dynamic Data Modeling Using Simultaneous Customer–Product Clustering Based on Markov Chains

ABSTRACT

This study aimed to develop and evaluate a dynamic data-modeling framework based on simultaneous customer–product clustering and Markov chains for identifying latent transactional structures, estimating temporal transitions, and improving prediction of evolving customer–product interactions. This quantitative, applied, longitudinal study was conducted using electronic transaction records from selected retail and online businesses in Tehran, Iran. The final analytical dataset included 1,200 customers and 480 products observed over 12 consecutive monthly periods. Customer–product interaction matrices were constructed from standardized measures of purchase frequency and transaction value. Simultaneous co-clustering was applied to identify joint customer and product groups, and alternative clustering configurations were compared using the silhouette coefficient, within-cluster dispersion, adjusted Rand stability, and reconstruction error. First-order discrete-time Markov chains were then used to estimate monthly transition probabilities among customer and product clusters. Predictive performance was evaluated using holdout validation and compared with static independent clustering, static co-clustering, and dynamic clustering without simultaneous customer–product modeling. The four-customer-cluster and four-product-cluster solution provided the strongest overall clustering structure, with the highest silhouette coefficient (0.562) and adjusted Rand stability index (0.847). Markov analysis showed substantial state persistence, with customer self-transition probabilities ranging from 0.694 to 0.805 and product self-transition probabilities ranging from 0.681 to 0.754. The proposed dynamic co-clustering model achieved customer and product cluster prediction accuracies of 0.872 and 0.849, respectively, with adjusted Rand indices of 0.793 and 0.768. It also produced the lowest prediction errors (MAE = 0.074; RMSE = 0.118), reducing RMSE by approximately 32.2% relative to static independent clustering and 21.9% relative to static simultaneous co-clustering. Integrating simultaneous customer–product clustering with Markov transition modeling provides a robust approach for representing both relational structure and temporal evolution in transactional data and improves predictive accuracy compared with static or independently modeled alternatives.

Keywords: Dynamic data modeling; customer–product co-clustering; Markov chains; customer segmentation; product clustering; transition probabilities; predictive analytics.

Introduction

The rapid expansion of digital commerce, online retailing, customer relationship management systems, and transaction-processing infrastructures has transformed the volume, velocity, and structural complexity of customer-related data. Modern organizations no longer interact with customers through isolated purchase events; rather, they accumulate longitudinal streams of transactions that simultaneously reflect changes in customer preferences, product demand, purchasing intensity, and market structure. These data are inherently dynamic because customers alter their purchasing patterns over time, products move through different stages of popularity and demand, and the relationships connecting customers to products

continuously evolve. Conventional analytical approaches that treat customer or product characteristics as temporally fixed may therefore fail to represent the true structure of transactional environments. Dynamic modeling methods have increasingly been used in different fields to describe systems whose states change over time, including electricity consumption behavior, infrastructure reliability, mobility, biological dynamics, and spatial processes, demonstrating the importance of explicitly incorporating temporal transitions into data analysis [1-4]. In commercial settings, this issue is especially important because the strategic value of a customer segment or product category depends not only on its current characteristics but also on the probability that it will evolve into another behavioral state.

Customer segmentation has traditionally been one of the principal analytical tools used to transform heterogeneous consumer data into actionable managerial knowledge. By dividing customers into relatively homogeneous groups according to purchase frequency, monetary value, behavioral characteristics, or preferences, organizations can design differentiated marketing strategies, optimize resource allocation, and improve customer retention. However, many conventional clustering techniques generate static partitions, implicitly assuming that customers remain associated with stable segments throughout the analytical period. Such an assumption is frequently unrealistic in transactional markets. Longitudinal segmentation research has shown that customer groups can change meaningfully over time and that dynamic segmentation can provide information that static classifications overlook [5]. Similarly, research combining clustering and state-transition models has demonstrated that customers may be assigned to meaningful segments while their subsequent movements among these segments are represented probabilistically [6]. These developments suggest that customer segmentation should be regarded not merely as a classification problem but as a temporal state-estimation problem in which both present membership and future transitions are of analytical interest.

A related challenge is that customers and products form an interdependent system. Customer behavior is observed through interactions with products, while product-level demand patterns are generated through interactions with customers. Nevertheless, many segmentation frameworks cluster customers and products independently. Such independent treatment may discard important relational information because a customer segment can only be fully understood in terms of the products that define its purchasing pattern, just as a product cluster is partly characterized by the types of customers who purchase it. Relational and graph-based analytical perspectives increasingly recognize that meaningful structures arise from interactions among entities rather than from isolated entity attributes. Dynamic relationship assessment using graph representation learning, for example, has illustrated how evolving relational information can be used to characterize heterogeneous user-generated interactions [7]. In other domains, stochastic block modeling has shown that latent groups and transition structures can be analyzed jointly when observations are linked through a network or interaction matrix [8]. These approaches support the broader methodological proposition that simultaneous customer–product clustering can uncover patterns that are difficult to identify when customers and products are modeled separately.

Simultaneous clustering, often referred to as co-clustering or biclustering, is particularly suitable for transaction matrices in which rows correspond to customers, columns correspond to products, and matrix entries represent interaction intensity. Instead of identifying groups along only one dimension, simultaneous clustering partitions both dimensions in a coordinated manner. The resulting co-clusters can reveal combinations of customer groups and product groups characterized by concentrated purchasing activity, weak interaction, high transaction value, or stable demand. Such structures are valuable because they provide a relational description of the market rather than a one-sided categorization. More broadly, clustering

methods have been employed in dynamic energy consumption analysis to group temporally varying behaviors [1], in phenotype estimation to construct interpretable data-driven states from complex longitudinal observations [9], and in spatial risk estimation to identify meaningful clusters under structured dependence [10]. The common principle across these applications is that latent groups become more informative when they are extracted from multidimensional data structures that preserve relevant dependencies among observations.

A second methodological requirement concerns temporal evolution. Even when a meaningful co-clustering structure has been identified, a static representation cannot explain how customers or products move among clusters over time. Markov chains provide a natural mathematical framework for modeling such transitions. A Markov chain represents a stochastic process in which the probability of a future state is modeled as a function of the current state. This structure is useful when the principal analytical objective is to estimate persistence, migration, or transition among a finite set of behavioral categories. Markov models have been applied to diverse dynamic systems, including telecommunication network downtime [11], malware and intrusion detection [12], computing-system availability [2], mobility transitions [4], and temperature-dependent biological dynamics [3]. Across these contexts, transition matrices offer a compact representation of system dynamics by quantifying the probability of remaining in the same state or moving to an alternative state.

The application of Markov models to customer behavior is especially appropriate because many purchasing processes display state dependence. A customer with low current engagement may be more likely to remain in a low-engagement state or move gradually toward moderate activity than to transition immediately into a high-value segment. Likewise, products with moderate demand may migrate into high-demand or low-demand states depending on changing customer preferences, promotions, seasonality, or market competition. Hidden Markov models have been used to characterize latent customer behavior when observed transactions are viewed as manifestations of underlying states [13]. Digital marketing research has also emphasized both the advantages and limitations of Markov-chain approaches for predicting e-commerce sales and interpreting customer movement through sequential stages [14]. More recently, digital customer journey research has highlighted the importance of sequential behavioral modeling for understanding how customers move through complex interaction pathways across time [15]. Together, these studies demonstrate that customer behavior is fundamentally sequential and that static snapshots provide only a partial account of purchasing dynamics.

The relevance of transition-based modeling is further reinforced by advances in longitudinal classification. Time-dependent transition rates can reveal latent heterogeneity that is not observable through ordinary cross-sectional classification procedures [16]. Similarly, nearly reducible finite Markov chains provide a theoretical basis for understanding systems in which observations show strong persistence within subsets of states but occasional transitions among them [17]. Such a pattern is highly plausible in customer–product environments, where most customers may remain behaviorally stable over short periods while a smaller proportion migrate among segments. Likewise, product categories may exhibit extended periods of stable demand interrupted by shifts caused by market trends or changes in customer interest. The ability of a Markov framework to distinguish between high self-transition probabilities and meaningful off-diagonal migration probabilities makes it particularly useful for differentiating stable from volatile clusters.

Despite these advantages, the estimation of dynamic states also raises computational and methodological challenges. State assignments may be sensitive to clustering initialization, model specification, the number of latent groups, or local optimization. Computational methods developed for other stochastic and high-dimensional systems illustrate the importance

of robust optimization and repeated estimation. Adaptive simulated annealing and tempering methods, for example, have been proposed for complex combinatorial optimization problems where conventional search procedures may become trapped in suboptimal solutions [18]. Machine-learning frameworks have likewise been applied to complex propagation processes in order to extract patterns from large-scale networked systems [19]. In addition, adaptable analytical platforms designed for high-dimensional dynamic data emphasize the importance of flexible preprocessing, visualization, and iterative model evaluation [20]. These methodological developments underline the need for clustering procedures that are evaluated not only in terms of fit but also in terms of stability, reproducibility, and predictive performance.

Theoretical developments in stochastic dynamics also contribute to the understanding of complex temporal clustering systems. The Swendsen–Wang family of stochastic processes has been studied as a mechanism for representing transitions among high-dimensional configurations, including dynamics on trees and systems with external fields [21, 22]. Although these models arise from a different application domain, they illustrate the broader principle that transitions among structured states can be studied through probabilistic dynamics rather than deterministic reclassification. Similarly, reinforcement-based stochastic models such as the rescaled Pólya urn show how local reinforcement can generate evolving probability structures over time [23]. These perspectives are relevant conceptually because customer–product interaction patterns may also exhibit reinforcement: repeated purchases can strengthen future purchase probabilities, while repeated exposure to specific product categories may increase the likelihood of continued engagement.

Temporal modeling has also been used to study systems in which dynamic change is coupled with spatial or structural dependence. Research on carbon-emission intensity has examined regional differences, dynamic evolution, and spillover effects simultaneously [24], while recent studies of agricultural efficiency and food security have incorporated spatial–temporal evolution and predictive modeling into a unified analytical framework [25]. Static and dynamic comparisons have likewise been used in transportation research to distinguish persistent structural characteristics from time-varying severity patterns [26]. These applications demonstrate that separating static structure from dynamic evolution can provide a more precise explanation of how complex systems change. In the context of customer–product analytics, an analogous distinction can be made between the contemporaneous co-cluster structure of the transaction matrix and the temporal transitions that connect those structures across successive periods.

Another important issue concerns the predictability of customer and product transitions. Predictive models are useful only when latent states display sufficient continuity to support forecasting. Recent research has demonstrated that sequential and state-based approaches can be used to estimate future states under conditions of partial or incomplete information, such as assigning load profiles when full smart-meter data are unavailable [27]. The same principle applies to customer analytics: organizations often possess incomplete histories for some customers, irregular product demand records, or sparse interaction matrices. A robust model should therefore derive predictive value from both the current relational structure and the observed transition behavior. Research on search strategies in digital environments has also shown that consumer decision processes are shaped by evolving sets of brand considerations rather than by isolated choices [28], further supporting the idea that customer–product engagement unfolds as a sequence of connected states.

The broader digitalization of supply chains and commercial ecosystems further strengthens the need for integrated dynamic models. Changes in digital infrastructure, data-sharing systems, and emerging technologies can alter the timing and structure of interactions among customers, firms, and products. Recent work on blockchain adoption and evolving supply-

chain structures has emphasized the importance of modeling adoption dynamics and system-level transitions under technologically changing conditions [29]. Social and interaction networks similarly exhibit dynamic segregation and connectivity patterns that can be represented using Markov bridge models [30]. These developments suggest that dynamic transactional systems should be analyzed as interconnected structures whose components co-evolve rather than as independent entities.

At the same time, state-transition models should not be interpreted as universally sufficient. Markov assumptions simplify complex behavior by representing future states primarily through current-state information. Although this simplification can make models tractable and interpretable, it may overlook longer historical dependencies, exogenous shocks, or nonlinear effects. The appropriateness of the Markov assumption therefore depends on the temporal resolution of the data and the empirical persistence of the inferred states. Work on Markovian representations of cortical spatiotemporal dynamics illustrates how complex temporal systems can sometimes be summarized through state-transition structures while still requiring careful validation of the underlying representation [31]. Similarly, high-dimensional state-transition systems benefit from methods that assess whether the identified states are sufficiently stable and meaningful to support predictive inference.

Taken together, previous research has established the value of clustering for identifying heterogeneous groups, Markov processes for representing temporal transitions, and relational approaches for preserving interactions among linked entities. However, an important methodological gap remains in integrating these components specifically for dynamic customer–product transaction data. Existing customer segmentation approaches often focus on customers alone, whereas product clustering is frequently conducted independently. Markov-based customer models, in turn, generally operate on predefined or separately derived customer states without simultaneously modeling the products through which those states are expressed. Consequently, there is a need for an integrated framework in which customer and product clusters are estimated jointly from transaction matrices and their temporal evolution is subsequently represented through transition probabilities. Such a framework can distinguish persistent from volatile customer segments, identify stable and changing product groups, estimate likely movements among states, and improve forecasting of future customer–product interactions.

Accordingly, the aim of the present study was to develop and evaluate a dynamic data-modeling framework based on simultaneous customer–product clustering and Markov chains to identify latent transactional structures, estimate temporal transitions among customer and product clusters, and improve prediction of evolving customer–product interaction patterns.

Methodology

This study employed a quantitative, applied, longitudinal data-mining design to develop and evaluate a dynamic model of customer–product relationships using simultaneous customer–product clustering and Markov chains. The study population consisted of customers of selected retail and online commercial businesses operating in Tehran, Iran, whose transaction histories were recorded electronically during a 12-month observation period. Following data screening and application of the eligibility criteria, transaction records belonging to 1,200 customers were included in the final analytical dataset. Customers were eligible for inclusion if they had completed at least three recorded purchase transactions during the observation period and if their transaction records contained valid information regarding customer identifier, product identifier, transaction date, quantity purchased, and transaction value. Customers with incomplete identification codes, duplicated profiles, anomalous transaction patterns attributable to system errors, or insufficient longitudinal records were excluded. The final

database contained transactions involving 480 distinct products distributed across the major product categories offered by the participating businesses. To preserve the temporal structure required for dynamic modeling, the observation period was divided into 12 consecutive monthly intervals, and a customer–product interaction matrix was constructed separately for each interval. Each matrix represented the purchasing relationships between the same set of customers and products at a specific point in time, thereby enabling the investigation of changes in customer clusters, product clusters, and customer–product associations across successive periods. All customer identifiers were anonymized prior to analysis, and no personally identifiable information was incorporated into the modeling process.

Data were collected using electronic transaction records extracted from the customer relationship management and sales information systems of the selected businesses. A structured data extraction framework was developed to ensure consistency in the collection and organization of the transactional information. The extracted variables included anonymous customer identification code, product identification code, transaction date, transaction frequency, quantity purchased, monetary value of each transaction, total expenditure, and product category. These variables were used to construct the longitudinal customer–product interaction database. Customer–product relationships were quantified using a weighted interaction measure reflecting both purchase frequency and transaction value, so that repeated and economically important interactions received greater representation than isolated purchases. Prior to model estimation, the raw transaction records were subjected to a systematic preprocessing procedure that included duplicate removal, consistency checking, treatment of missing observations, detection of extreme values, aggregation of repeated transactions within each temporal interval, and normalization of variables measured on different scales. Continuous transaction variables were normalized to prevent variables with larger numerical magnitudes from disproportionately influencing the clustering process. For each month, an interaction matrix was generated in which rows represented customers, columns represented products, and matrix elements represented the standardized intensity of interaction between each customer and product. These matrices constituted the principal input to the simultaneous clustering algorithm. In addition, a transition matrix was generated for each pair of consecutive periods to capture movements of customers and products between the clusters identified at adjacent time points. The resulting data structure therefore incorporated both cross-sectional customer–product relationships and temporal changes in cluster membership.

Data analysis was conducted through an integrated procedure combining simultaneous customer–product clustering with Markov-chain-based dynamic modeling. In the first analytical stage, the customer–product interaction matrix for each temporal period was subjected to simultaneous clustering, also referred to as co-clustering or biclustering, in which customers and products were partitioned jointly rather than being clustered independently. The objective of this procedure was to identify internally homogeneous customer groups characterized by similar purchasing patterns while simultaneously identifying product groups exhibiting similar patterns of customer demand. The clustering algorithm iteratively optimized customer and product assignments by minimizing within-cluster dissimilarity and maximizing between-cluster separation in the customer–product interaction structure. Candidate solutions with different numbers of customer and product clusters were evaluated, and the final cluster configuration was selected on the basis of clustering quality, stability, and interpretability. Internal validation indices, including the silhouette coefficient and within-cluster dispersion, were employed to assess the adequacy of alternative clustering structures. Cluster stability was additionally examined across repeated initializations to reduce the possibility that the final configuration resulted from local optimization.

After obtaining the simultaneous customer and product clusters for each monthly interval, the temporal evolution of cluster membership was modeled using first-order discrete-time Markov chains. Separate transition probability matrices were estimated for customer clusters and product clusters. Each element of a transition matrix represented the conditional probability that an observation assigned to a particular cluster at time (t) would remain in the same cluster or move to another cluster at time ($t+1$). Transition probabilities were estimated by dividing the number of observed transitions from each origin state to a destination state by the total number of observations originating in that state. The diagonal elements of each matrix therefore represented cluster persistence probabilities, whereas off-diagonal elements reflected migration between behavioral states. This approach made it possible to distinguish stable customer and product segments from clusters characterized by substantial temporal mobility. The Markov property was assessed by examining whether the subsequent cluster state could be adequately represented as a function of the current state, while transition patterns across successive intervals were compared to evaluate temporal consistency.

The simultaneous clustering and Markov components were subsequently integrated into a unified dynamic framework. For each period, customer–product co-clusters were treated as the structural representation of the market, whereas transition probabilities described the temporal evolution of these structures. The model was used to estimate the probability distribution of future customer and product cluster memberships and to reconstruct expected customer–product interaction patterns in subsequent periods. Model performance was evaluated using a holdout forecasting procedure in which earlier monthly observations were used for model estimation and the final temporal intervals were reserved for validation. Predicted customer and product cluster memberships were compared with observed memberships using classification accuracy and adjusted Rand index, while the reconstruction accuracy of the customer–product interaction matrices was assessed using mean absolute error and root mean square error. The predictive performance of the proposed dynamic co-clustering model was also compared with static clustering and clustering models without Markov transitions in order to determine whether explicitly modeling temporal dependence improved predictive accuracy. All analyses were performed using Python, with appropriate libraries for numerical computation, machine learning, clustering, Markov transition estimation, and model validation. Statistical and computational decisions were evaluated at a significance level of 0.05 where inferential testing was applicable, and multiple random initializations were used for optimization-based procedures to improve the robustness and reproducibility of the results.

Findings and Results

The final analytical sample consisted of 1,200 customers whose transaction histories satisfied the predefined inclusion criteria and remained sufficiently complete for longitudinal analysis across the 12-month observation period. Of the customers included in the study, 648 (54.0%) were male and 552 (46.0%) were female. The mean age of customers was 36.84 years ($SD = 9.71$), with ages ranging from 18 to 68 years. Regarding age distribution, 234 customers (19.5%) were younger than 25 years, 416 (34.7%) were aged 25–34 years, 338 (28.2%) were aged 35–44 years, and 212 (17.7%) were 45 years or older. In terms of purchase activity, customers completed an average of 18.73 transactions during the study period ($SD = 9.46$), with a median of 16 transactions. The average number of distinct products purchased per customer was 21.38 ($SD = 10.52$), indicating substantial heterogeneity in purchasing breadth across the sample. The final database included 480 unique products belonging to the major product categories available in the participating businesses. Across the entire observation

period, 22,476 eligible purchase transactions were analyzed. The mean standardized monthly customer–product interaction intensity was 0.384 (SD = 0.217). Preliminary inspection of the transaction records indicated sufficient temporal variability in customer purchasing patterns and product demand to justify the use of dynamic rather than exclusively static clustering techniques.

Table 1

Descriptive Characteristics of Customer–Product Transaction Variables

Variable	Mean	SD	Minimum	Maximum	Median
Number of transactions per customer	18.73	9.46	3	61	16.00
Number of distinct products purchased	21.38	10.52	3	74	19.00
Monthly purchase frequency	1.56	0.79	0.25	5.08	1.33
Standardized transaction value	0.417	0.231	0.012	1.000	0.386
Standardized customer–product interaction intensity	0.384	0.217	0.008	1.000	0.351
Product purchase frequency	46.83	31.74	6	198	38.00
Product demand stability index	0.672	0.184	0.141	0.958	0.701
Customer behavioral stability index	0.638	0.201	0.109	0.947	0.664

As shown in Table 1, the transaction data exhibited considerable variation across customers and products. Customers completed an average of 18.73 transactions during the 12-month period, whereas the mean number of distinct products purchased was 21.38, demonstrating that many customers interacted with a relatively broad set of products rather than repeatedly purchasing only a small number of items. The average monthly purchase frequency was 1.56 transactions, although the maximum value of 5.08 reflected a subgroup of highly active customers. The mean standardized transaction value was 0.417, and the corresponding mean customer–product interaction intensity was 0.384, indicating a moderately sparse interaction structure in which most customer–product pairs were characterized by relatively low or no interaction, while a smaller subset demonstrated substantially stronger purchasing relationships. Product purchase frequency varied markedly, ranging from 6 to 198 purchases, with a mean of 46.83, confirming heterogeneity in product popularity. The product demand stability index averaged 0.672, suggesting that although a substantial proportion of products maintained relatively stable demand patterns, a meaningful share experienced temporal changes in purchasing intensity. Similarly, the mean customer behavioral stability index was 0.638, indicating that customer purchasing behavior was moderately persistent but not temporally invariant. These characteristics supported the application of a dynamic co-clustering model capable of representing both stable customer–product structures and transitions between behavioral states.

Table 2

Evaluation of Alternative Simultaneous Customer–Product Clustering Solutions

Customer Clusters	Product Clusters	Silhouette Coefficient	Within-Cluster Dispersion	Adjusted Rand Stability	Mean Reconstruction RMSE
3	3	0.481	0.294	0.782	0.163
3	4	0.503	0.281	0.798	0.154
4	3	0.517	0.268	0.816	0.148
4	4	0.562	0.241	0.847	0.131
4	5	0.548	0.236	0.834	0.129
5	4	0.539	0.229	0.821	0.128
5	5	0.526	0.221	0.803	0.126
6	5	0.491	0.214	0.771	0.124

Table 2 presents the comparative evaluation of candidate co-clustering configurations. Although increasing the number of clusters generally reduced within-cluster dispersion and reconstruction error, solutions containing a larger number of clusters exhibited lower stability and weaker separation. The four-customer-cluster and four-product-cluster solution

produced the highest silhouette coefficient (0.562) and the highest adjusted Rand stability index (0.847), while maintaining a relatively low reconstruction RMSE of 0.131. The four-by-five and five-by-four solutions yielded slightly lower reconstruction error, but this marginal improvement was accompanied by reductions in cluster separation and stability. Accordingly, the four-by-four configuration was selected as the optimal solution because it provided the most favorable balance among internal cohesion, between-cluster separation, reproducibility across random initializations, reconstruction accuracy, and substantive interpretability. In the selected solution, the four customer clusters represented distinct behavioral purchasing patterns, while the four product clusters reflected differences in demand intensity, purchase regularity, and customer concentration. The first customer cluster contained 328 customers and was characterized by relatively low transaction frequency and low interaction intensity. The second cluster included 296 customers with moderate purchase frequency and relatively broad product diversity. The third cluster comprised 341 customers with high purchase frequency and stronger interactions with frequently purchased products. The fourth cluster contained 235 customers and represented a comparatively smaller but highly valuable and behaviorally persistent group. Product clusters similarly differed in their demand profiles, with the first product cluster representing low-frequency products, the second containing moderately stable products, the third containing high-demand products, and the fourth representing products with high transaction values but greater temporal variability. These results indicated that simultaneous clustering captured interpretable latent structures that would have been obscured if customers and products had been classified independently.

Table 3

Estimated Monthly Markov Transition Probability Matrix for Customer Clusters

Origin Cluster	Customer Cluster 1	Customer Cluster 2	Customer Cluster 3	Customer Cluster 4
Customer Cluster 1	0.741	0.173	0.062	0.024
Customer Cluster 2	0.109	0.694	0.154	0.043
Customer Cluster 3	0.041	0.132	0.748	0.079
Customer Cluster 4	0.018	0.061	0.116	0.805

As reported in Table 3, the diagonal probabilities were consistently higher than the off-diagonal probabilities, demonstrating substantial persistence in customer cluster membership from one month to the next. Customer Cluster 4 showed the greatest stability, with a probability of 0.805 that a customer in this cluster would remain in the same behavioral state during the subsequent period. Customer Cluster 3 also exhibited a high persistence probability of 0.748, followed by Customer Cluster 1 at 0.741 and Customer Cluster 2 at 0.694. The comparatively lower stability of Customer Cluster 2 suggested that customers with moderate purchasing intensity were more likely to transition to adjacent behavioral states than customers at either the lower or higher ends of the purchasing spectrum. The most frequent transition from Customer Cluster 1 was toward Customer Cluster 2, with a probability of 0.173, indicating that low-activity customers were more likely to move gradually toward moderate purchasing activity than to shift directly into high-value states. Similarly, the probability of transition from Cluster 2 to Cluster 3 was 0.154, whereas the probability of transition directly from Cluster 2 to Cluster 4 was only 0.043. Customers in Cluster 3 had a probability of 0.079 of progressing to Cluster 4 and a probability of 0.132 of moving backward to Cluster 2. For Customer Cluster 4, downward movement was relatively uncommon, although the most likely transition was to Cluster 3 with a probability of 0.116. These transition patterns supported a predominantly gradual behavioral evolution process in which customers tended to remain in their existing state or shift toward an adjacent cluster rather than move abruptly between highly dissimilar purchasing states.

Table 4*Markov Transition Probability Matrix for Product Clusters and Long-Run State Distribution*

Origin Product Cluster	Product Cluster 1	Product Cluster 2	Product Cluster 3	Product Cluster 4	Long-Run Probability
Product Cluster 1	0.718	0.184	0.069	0.029	0.234
Product Cluster 2	0.126	0.681	0.149	0.044	0.261
Product Cluster 3	0.047	0.137	0.752	0.064	0.301
Product Cluster 4	0.032	0.073	0.141	0.754	0.204

Table 4 demonstrates that product clusters also exhibited marked temporal persistence. Products in Cluster 3 showed the highest probability of remaining in the same state across successive months (0.752), closely followed by Product Cluster 4 (0.754), Product Cluster 1 (0.718), and Product Cluster 2 (0.681). Similar to the customer transition structure, product movements occurred predominantly between adjacent clusters. Products classified in the low-demand Cluster 1 had an 0.184 probability of moving to Cluster 2 but only an 0.029 probability of moving directly to Cluster 4. Product Cluster 2 displayed greater mobility, with a 0.149 probability of moving toward Cluster 3 and an 0.126 probability of returning to Cluster 1. Products in Cluster 3 demonstrated a probability of 0.137 of moving to Cluster 2 and 0.064 of moving to Cluster 4. Products in Cluster 4 were relatively stable but had a 0.141 probability of moving into Cluster 3 in the subsequent period. Examination of the estimated long-run distribution indicated that Product Cluster 3 would account for the largest equilibrium share of products, with a stationary probability of 0.301, followed by Product Cluster 2 with 0.261, Product Cluster 1 with 0.234, and Product Cluster 4 with 0.204. These findings imply that the market structure was not expected to converge toward either exclusively low-demand or exclusively high-value products. Instead, the Markov process suggested a long-run configuration dominated by moderately high and relatively stable demand states, reflecting a persistent but dynamically evolving product portfolio.

Table 5*Predictive Performance of the Proposed Dynamic Co-Clustering Model Compared with Benchmark Models*

Model	Customer Cluster Accuracy	Product Cluster Accuracy	Customer ARI	Product ARI	MAE	RMSE
Static independent clustering	0.724	0.701	0.612	0.587	0.119	0.174
Static simultaneous co-clustering	0.781	0.762	0.681	0.655	0.103	0.151
Dynamic clustering without simultaneous customer–product modeling	0.813	0.794	0.726	0.701	0.091	0.139
Proposed dynamic co-clustering with Markov transitions	0.872	0.849	0.793	0.768	0.074	0.118

As shown in Table 5, the proposed dynamic simultaneous customer–product clustering model with Markov transitions achieved the strongest predictive performance across all evaluation indices. Customer cluster prediction accuracy reached 0.872, compared with 0.813 for dynamic clustering without simultaneous customer–product modeling, 0.781 for static co-clustering, and 0.724 for static independent clustering. A similar pattern was observed for product cluster prediction, for which the proposed model achieved an accuracy of 0.849. The adjusted Rand index for predicted customer cluster membership was 0.793, indicating strong correspondence between predicted and observed customer partitions, while the product adjusted Rand index reached 0.768. Both values exceeded those of all benchmark approaches. Reconstruction performance also improved considerably when temporal transition information was incorporated. The proposed model yielded an MAE of 0.074 and an RMSE of 0.118, compared with an RMSE of 0.139 for dynamic clustering without simultaneous modeling, 0.151 for static co-clustering, and 0.174 for static independent clustering. Relative to static independent clustering,

the proposed method reduced RMSE by approximately 32.2%. Relative to static simultaneous co-clustering, the reduction was approximately 21.9%. These findings demonstrated that the predictive advantage of the proposed framework was not attributable solely to the use of co-clustering or solely to the incorporation of temporal information; rather, the combination of simultaneous customer–product structure and Markov transition dynamics produced the highest level of predictive accuracy. The results further suggested that ignoring temporal dependence led to loss of information concerning state persistence and inter-cluster mobility, whereas modeling customers and products separately failed to capture the reciprocal structure of the underlying transaction matrix.

Figure 1

Temporal Evolution of Customer and Product Cluster Membership Across the 12-Month Observation Period

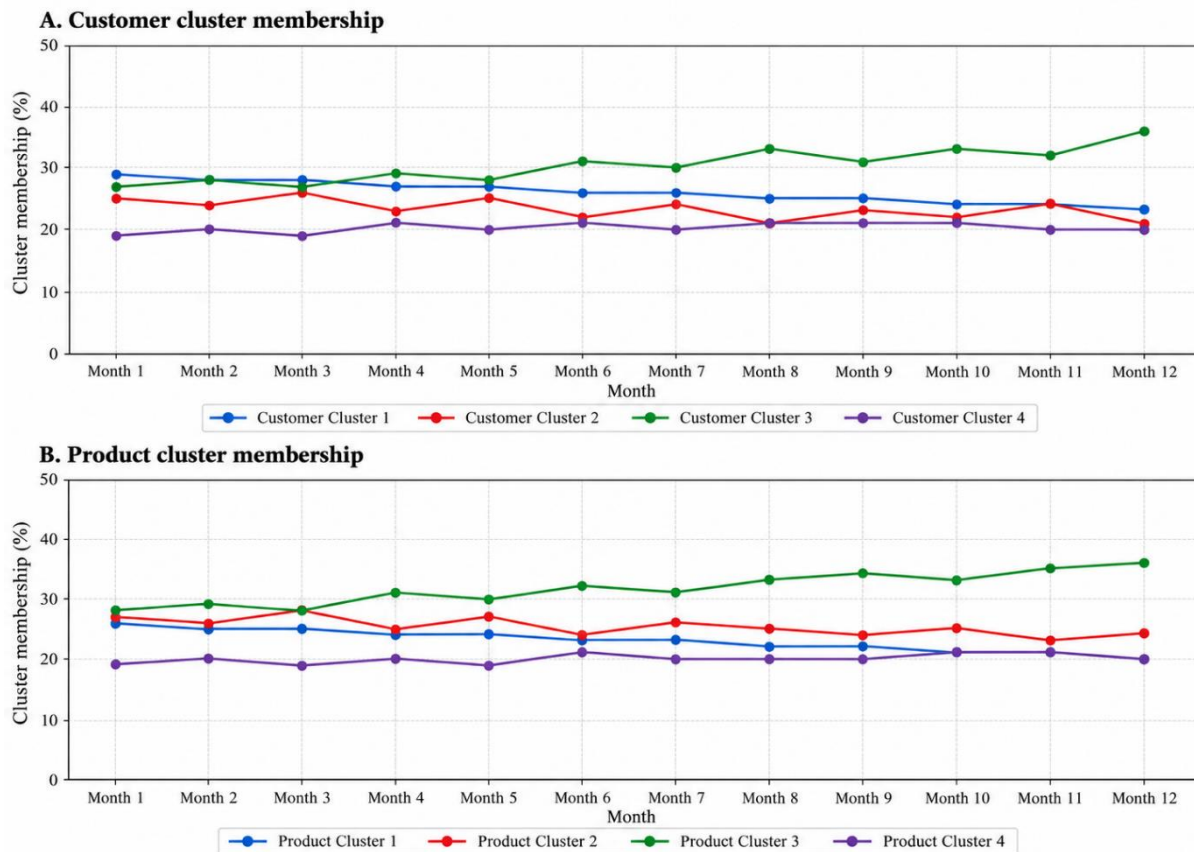


Figure 1 illustrates the evolution of the four customer clusters and four product clusters across the 12 successive monthly periods. The temporal profiles showed that cluster proportions remained broadly stable while still demonstrating meaningful month-to-month movements. Customer Cluster 4 maintained the smallest but most persistent share throughout the observation period, consistent with its high self-transition probability. Customer Cluster 2 displayed the greatest variation in membership size, reflecting the relatively high rates of transition both toward lower-activity Cluster 1 and toward higher-activity Cluster 3. Customer Cluster 3 gradually increased in relative size during the middle and later observation periods, whereas Cluster 1 showed a modest decline, suggesting a net movement of some customers toward more intensive purchasing states. Product clusters exhibited a comparable pattern of gradual rather than abrupt structural change. Product Cluster 3 remained the largest high-demand state for most of the observation period and increased moderately toward the final months, while Product Cluster 2 fluctuated more strongly. Cluster 4 products demonstrated relatively stable

membership despite variability in transaction intensity, and Cluster 1 declined modestly over the study period. Overall, the temporal pattern displayed in Figure 1 was consistent with the estimated Markov transition matrices and indicated that the customer–product system was characterized by simultaneous persistence and controlled mobility rather than random reassignment across periods.

Taken together, the findings provided consistent evidence that customer and product behaviors were organized into identifiable and temporally persistent latent states. The optimal four-by-four co-clustering structure achieved strong internal separation and reproducibility, while the estimated Markov matrices revealed that both customer and product states were characterized by high self-transition probabilities and predominantly local movements between neighboring clusters. The predictive analysis further demonstrated that incorporating temporal dynamics into simultaneous customer–product clustering substantially improved out-of-sample classification and reconstruction performance. The proposed model therefore captured not only the contemporaneous organization of customer–product relationships but also the manner in which these relationships evolved across time, providing a more complete representation of dynamic transactional behavior than static or independently clustered alternatives.

Discussion and Conclusion

The present study developed and evaluated a dynamic data-modeling framework based on simultaneous customer–product clustering and Markov chains and examined whether integrating relational clustering with temporal transition modeling could provide a more accurate representation of transactional behavior than static or independently clustered alternatives. The findings showed that the four-customer-cluster and four-product-cluster solution provided the strongest overall clustering structure, with the highest silhouette coefficient and adjusted Rand stability index among the candidate configurations. The results also demonstrated substantial temporal persistence in both customer and product states, as reflected in high diagonal probabilities in the corresponding Markov transition matrices. In addition, transitions generally occurred between behaviorally adjacent clusters rather than between highly dissimilar states. Finally, the proposed dynamic co-clustering model achieved superior predictive performance compared with static independent clustering, static simultaneous co-clustering, and dynamic clustering without simultaneous customer–product modeling. Collectively, these findings support the central premise that customer and product behaviors should be modeled jointly and dynamically rather than as isolated and temporally fixed structures.

One of the principal findings was that a four-by-four simultaneous clustering structure provided the most favorable balance of cluster separation, stability, reconstruction accuracy, and interpretability. This result indicates that the customer–product transaction matrix contained a meaningful latent structure in which customer groups and product groups could be jointly distinguished according to patterns of interaction intensity, frequency, value, and demand stability. The selected model did not simply minimize reconstruction error; rather, it retained stronger separation and reproducibility than more fragmented solutions with greater numbers of clusters. This finding is important because excessive segmentation can produce unstable or operationally ambiguous groups even when numerical fit improves marginally. Similar concerns have been raised in dynamic classification studies, where latent grouping must reflect both empirical heterogeneity and stable transition structures [16]. The result is also broadly consistent with dynamic clustering research demonstrating that meaningful segmentation depends on preserving the temporal and structural organization of the data rather than maximizing the number

of identified categories [1]. From this perspective, the four-by-four solution appears to represent a parsimonious decomposition of the customer–product system into a manageable set of relational states.

The superiority of simultaneous clustering over independent clustering further suggests that customer behavior cannot be understood adequately without considering the products through which that behavior is expressed. The reciprocal relationship between customers and products means that a customer cluster acquires meaning partly from the specific products with which its members interact, while a product cluster is defined partly by the types of customers generating its demand. This interpretation is consistent with relational and graph-based approaches that emphasize modeling interactions rather than treating units as isolated observations [7]. It is also conceptually aligned with stochastic block modeling, in which latent structures are inferred from interaction patterns among connected entities [8]. In the present study, simultaneous clustering captured these dependencies directly, which likely explains why static co-clustering outperformed static independent clustering in both cluster accuracy and matrix reconstruction. The finding therefore reinforces the methodological value of preserving the two-mode structure of transactional data.

The customer transition matrix revealed that all four customer clusters exhibited relatively high self-transition probabilities, with the greatest persistence observed for Customer Cluster 4. This cluster represented customers with high transaction intensity, higher value, and greater behavioral stability, and its self-transition probability of 0.805 indicated that these customers were highly likely to remain in the same state from one month to the next. Customer Cluster 3 also demonstrated substantial persistence, while the intermediate Customer Cluster 2 showed the greatest mobility. This pattern suggests that customers located at the more behaviorally defined ends of the segmentation structure are relatively stable, whereas customers with moderate purchasing behavior are more susceptible to movement between adjacent states. Comparable interpretations have emerged from customer behavior models based on hidden Markov structures, where latent customer states differ in both persistence and transition propensity [13]. The current findings are also consistent with the customer segmentation approach of Harish and Malathy, which combined clustering and Markov modeling to represent both segment membership and movement among retail customer groups [6].

The predominantly local nature of the customer transitions is another notable result. Customers in the lowest-activity cluster were most likely to move into the moderate-activity cluster rather than transition directly into the highest-value cluster, while customers in Cluster 2 were more likely to move into Cluster 3 than Cluster 4. Similarly, downward movements generally occurred toward neighboring states. This pattern supports the interpretation of customer behavior as a gradual evolutionary process rather than an abrupt reclassification process. Sequential customer journey research has likewise emphasized that digital behaviors tend to develop through ordered pathways of interaction rather than disconnected events [15]. The current result also accords with research on evolving sets of brand consideration, which suggests that customer choices are influenced by cumulative and sequential decision processes rather than instantaneous switching among unrelated alternatives [28]. Accordingly, the Markov structure observed in the present study appears to capture a realistic form of behavioral continuity.

The product transition matrix demonstrated a similar pattern of state persistence combined with moderate mobility. Product Clusters 3 and 4 displayed particularly high probabilities of remaining in the same state, while Product Cluster 2 showed the greatest transition activity. These results imply that products with clearly established demand profiles, whether high and stable or high-value and specialized, tend to maintain their market positions over short intervals. In contrast,

moderately performing products may be more sensitive to fluctuations in demand and therefore more likely to migrate between states. This interpretation is consistent with the broader use of dynamic state models in systems where observed units move among severity, demand, or performance categories over time. For example, static and dynamic analyses of transportation delay severity have demonstrated that dynamic transition modeling can reveal patterns that remain concealed when observations are treated as temporally fixed [26]. Similarly, dynamic evolution research in regional systems has shown that states may exhibit both persistence and directional transition depending on structural conditions [24, 25].

The estimated long-run distribution of product states further suggested that the system would not converge toward a single dominant product condition. Instead, the stationary distribution placed the largest share in Product Cluster 3, followed by Product Clusters 2, 1, and 4. This result indicates that the market is likely to preserve heterogeneity over time, even when transition probabilities remain relatively stable. Such persistence of multiple states is compatible with the theory of nearly reducible Markov chains, in which systems may remain for extended periods within groups of related states while still allowing occasional transitions among them [17]. The finding also reflects the logic of stochastic systems in which long-run behavior emerges from repeated local transitions rather than deterministic convergence. Similar principles are evident in other stochastic dynamic models, including reinforced processes and high-dimensional Markov systems [21-23]. In practical terms, the result suggests that product portfolios should be expected to remain structurally diverse rather than becoming uniformly concentrated in a single demand category.

The strongest evidence in support of the proposed framework came from the comparative predictive analysis. The dynamic simultaneous co-clustering model with Markov transitions achieved the highest customer cluster accuracy, product cluster accuracy, customer adjusted Rand index, product adjusted Rand index, and the lowest MAE and RMSE. This result shows that both dimensions of the model contributed meaningful information. Static independent clustering performed worst, indicating that ignoring both relational dependence and temporal evolution imposed a substantial loss of predictive power. Static co-clustering improved performance by preserving the customer–product interaction structure, whereas dynamic clustering without simultaneous modeling produced a further improvement by incorporating temporal information. The best results were achieved only when both components were combined. This pattern strongly suggests that relational and temporal dependencies are complementary rather than substitutable sources of predictive information.

The improved predictive performance of the Markov component is consistent with a wide range of applications in which transition structures enhance forecasting. Discrete-time Markov chains have been effectively used to model telecommunication network downtime severity [11], system availability [2], mobility transitions [4], and biological dynamics [3]. These applications demonstrate that prediction improves when the current state is treated as informative about subsequent behavior. In the commercial domain, Markov chains have similarly been proposed as useful tools for forecasting e-commerce sales and digital marketing outcomes [14]. The current findings extend this logic by showing that transition modeling becomes even more informative when the underlying states are derived from a joint customer–product representation rather than from customers or products separately.

The findings also have implications for how sparse and incomplete transaction data may be handled. Transaction matrices are often characterized by a large proportion of zero or weak interactions, yet meaningful structure can still emerge when both entity dimensions are modeled together and transitions are estimated over time. Research on dynamic load-profile assignment has shown that useful state representations can be obtained even under partial observation when clustering and

predictive structure are appropriately integrated [27]. Likewise, machine-learning approaches have demonstrated the capacity to extract latent patterns from noisy and heterogeneous data environments [19, 32]. The relatively strong predictive accuracy observed in the present study suggests that co-clustering can partially address sparsity by pooling structurally similar customers and products, while the Markov component provides temporal regularization through persistence probabilities.

The results also support the broader argument that dynamic systems should be modeled using frameworks capable of adapting to evolving configurations. Flexible analysis environments and high-dimensional modeling approaches have increasingly emphasized iterative state estimation, repeated model evaluation, and structural adaptation [20, 31]. Similarly, adaptive optimization techniques have been developed to improve search performance in complex combinatorial spaces where naïve methods may become trapped in locally optimal configurations [18]. Although the present study did not directly employ all such approaches, its findings reinforce the value of considering stability and repeated optimization when constructing cluster-based dynamic models. The observed superiority of the selected four-by-four structure suggests that effective dynamic modeling requires balancing complexity against reproducibility.

From a theoretical perspective, the findings indicate that customer–product markets can be conceptualized as systems of latent relational states linked through probabilistic transitions. This interpretation is broader than conventional segmentation because the unit of analysis is not merely the customer or product but the evolving interaction structure connecting them. Such a view is consistent with studies of dynamic networks, evolving relational structures, and Markov bridge models in which system states are inferred through changing patterns of connectivity [30]. It is also compatible with emerging approaches to digitally mediated commercial systems in which technology, infrastructure, and transactional relationships co-evolve over time [29]. The empirical findings of the present study therefore support an integrated perspective in which market segmentation and transition modeling are treated as interconnected analytical tasks.

Overall, the study demonstrated that dynamic simultaneous customer–product clustering provides a more comprehensive representation of transactional behavior than static or one-sided alternatives. The identified clusters were stable enough to support meaningful interpretation but sufficiently dynamic to exhibit measurable transitions over time. The Markov matrices captured both persistence and mobility, while the comparative prediction results showed that incorporating these transition structures improved forecasting substantially. These findings confirm that customer and product states are neither fully stable nor randomly fluctuating; rather, they evolve through structured pathways that can be modeled probabilistically. The combination of co-clustering and Markov chains therefore appears to be a suitable framework for representing dynamic customer–product systems and for generating interpretable predictions about future transactional states.

The present study had several limitations. First, the analysis was based on transactional data from selected businesses in Tehran, which may limit the generalizability of the findings to other geographic, cultural, industrial, or platform contexts. Second, the observation period covered 12 months and may therefore have captured seasonal variation but not longer-term structural changes in customer preferences or product life cycles. Third, the model relied primarily on transactional indicators and did not incorporate potentially relevant contextual variables such as customer demographics, promotional exposure, competitor actions, macroeconomic conditions, website behavior, or product reviews. Fourth, the first-order Markov assumption treated the current cluster state as the principal determinant of the next state and therefore did not explicitly capture longer memory effects. Fifth, cluster assignments may remain sensitive to preprocessing choices, distance metrics, normalization procedures, and optimization settings despite the use of repeated initializations and stability assessment.

Finally, the proposed model was evaluated using one empirical dataset, and broader validation across independent datasets would strengthen conclusions regarding robustness.

Future studies are suggested to evaluate the proposed framework using larger and more heterogeneous datasets collected from multiple industries, regions, and digital platforms. Longer observation periods could be employed to distinguish short-term fluctuations from enduring structural transitions and to examine seasonality, trend, and product life-cycle effects more explicitly. Researchers may also extend the framework by incorporating higher-order Markov chains, hidden Markov models, semi-Markov processes, or nonhomogeneous transition probabilities that vary with external covariates. Future models could integrate demographic, contextual, behavioral, and marketing variables with transactional data and compare classical co-clustering with graph neural networks, tensor factorization, temporal embedding methods, and deep sequential architectures. It would also be useful to investigate whether transition probabilities differ systematically across product categories, customer value groups, or macroeconomic conditions and to assess model performance under varying levels of data sparsity and missingness. Replication using independent holdout datasets and prospective forecasting designs would provide stronger evidence regarding the generalizability of the framework.

For practice, organizations can use the proposed framework to monitor customer and product states continuously rather than relying on periodically updated static segmentation reports. Customer clusters with high upward transition probabilities can be targeted with retention and development strategies, while customers with elevated probabilities of moving toward lower-value states can be identified for early intervention. Product transition patterns can support inventory planning, demand forecasting, assortment management, and promotional allocation by identifying products likely to move into higher- or lower-demand states. Managers can also use the customer–product co-cluster structure to design more precise cross-selling and recommendation strategies because each customer group is linked directly to product groups with which it demonstrates strong interaction. Re-estimating the transition matrices at regular intervals would allow firms to detect structural market changes and update their decisions accordingly. Integrating this framework into customer relationship management and business intelligence systems could therefore provide a practical basis for more adaptive, predictive, and data-driven marketing and product-management decisions.

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Authors' Contributions

All authors equally contributed to this study.

Declaration of Interest

The authors of this article declared no conflict of interest.

Ethical Considerations

The study protocol adhered to the principles outlined in the Helsinki Declaration, which provides guidelines for ethical research involving human participants. Written consent was obtained from all participants in the study.

Transparency of Data

In accordance with the principles of transparency and open research, we declare that all data and materials used in this study are available upon request.

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References

- [1] Y. Zhang, Q. Ai, and Z. Li, "Grouping of Dynamic Electricity Consumption Behaviour: An α -divergence Based Hierarchical Clustering Model," *Iet Generation Transmission & Distribution*, vol. 15, no. 22, pp. 3164-3175, 2021, doi: 10.1049/gtd2.12248.
- [2] R. Paharsingh, "Availability Modeling of Computing Systems With Virtual Architectures," 2021, doi: 10.32920/ryerson.14661660.v1.
- [3] P. Damos, J. Dorrestijn, T. Thomidis, J. Tuells, and P. Caballero, "A Temperature Conditioned Markov Chain Model for Predicting the Dynamics of Mosquito Vectors of Disease," *Insects*, vol. 12, no. 8, p. 725, 2021, doi: 10.3390/insects12080725.
- [4] M. Dhont and E. Tsiorkova, "Elucidating Transition State Behaviour From Mobility Data by Cascades of Markov Chains," 2023, doi: 10.21428/594757db.d65bfe6f.
- [5] S. Lucas Lopes Ferreira de, F. Bassi, and F. Ana Augusta Ferreira de, "Longitudinal Analysis of Microfinance Borrowers in Brazil: A Dynamic Market Segmentation," *Journal of International Development*, vol. 33, no. 6, pp. 1063-1083, 2021, doi: 10.1002/jid.3561.
- [6] A. Harish and C. Malathy, "Customer Segment Prediction on Retail Transactional Data Using K-Means and Markov Model," *Intelligent Automation & Soft Computing*, vol. 36, no. 1, pp. 589-600, 2023, doi: 10.32604/iasc.2023.032030.
- [7] R. Huang, Z. Chen, J. He, and X. Chu, "Dynamic Heterogeneous User Generated Contents-Driven Relation Assessment via Graph Representation Learning," *Sensors*, vol. 22, no. 4, p. 1402, 2022, doi: 10.3390/s22041402.
- [8] Q. Duchemin, "Reliable Prediction in the Markov Stochastic Block Model," *Esaim Probability and Statistics*, vol. 27, pp. 80-135, 2023, doi: 10.1051/ps/2022019.
- [9] Y. Wang *et al.*, "A Methodology of Phenotyping ICU Patients From EHR Data: High-Fidelity, Personalized, and Interpretable Phenotypes Estimation," 2023, doi: 10.1101/2023.03.15.23287315.
- [10] M. Bani-Yaghoub, K. Rekab, J. Pluta, and S. Tabharit, "Estimating the Relative Risks of Spatial Clusters Using a Predictor–Corrector Method," *Mathematics*, vol. 13, no. 2, p. 180, 2025, doi: 10.3390/math13020180.
- [11] I. W. Abdul and I. A. Gedel, "Modeling Downtime Severity of Telecommunication Networks Using Discrete Time Markov Chains," *International Journal of Advanced Technology and Engineering Exploration*, vol. 10, no. 101, 2023, doi: 10.19101/ijatee.2022.10100435.
- [12] E. Nikolova, "Markov Models for Malware and Intrusion Detection: A Survey," *Serdica Journal of Computing*, vol. 15, no. 2, pp. 129-147, 2023, doi: 10.55630/sjc.2021.15.129-147.
- [13] A. Jandera and T. Škovránek, "Customer Behaviour Hidden Markov Model," *Mathematics*, vol. 10, no. 8, p. 1230, 2022, doi: 10.3390/math10081230.
- [14] L. Amalia, Sertiningsih, F. Hayatuki, S. Muliani, and A. M. Junaidi, "Advantages and Disadvantages of Utilizing Markov Chains in a Digital Marketing Strategy to Predict E-Commerce Sales," *Scientia Technology Science and Society*, vol. 1, no. 3, pp. 43-61, 2024, doi: 10.59324/stss.2024.1(3).03.
- [15] L. Bar-El, H. Chalutz-BenGal, S. Gazit, T. Patalon, and I. Ben-Gal, "Analyzing the Digital Customer Journey: A Novel Framework for Sequential Behavior Modeling," *Scientific Reports*, vol. 16, no. 1, 2026, doi: 10.1038/s41598-026-43762-8.
- [16] J. Chang and W. Chan, "Latent Classification of Time-Dependent Transition Rates in Longitudinal Binary Outcome Data," *Statistical Methods in Medical Research*, vol. 35, no. 1, pp. 61-78, 2025, doi: 10.1177/09622802251393610.

- [17] D. Sharpe and D. J. Wales, "Nearly Reducible Finite Markov Chains: Theory and Algorithms," *The Journal of Chemical Physics*, vol. 155, no. 14, 2021, doi: 10.1063/5.0060978.
- [18] S. Xiao, P. Pan, P. Zheng, and Z. Wu, "A Hybrid Adaptive Simulated Annealing and Tempering Algorithm for Solving the Half-Open Multi-Depot Vehicle Routing Problem," *Mathematics*, vol. 12, no. 7, p. 947, 2024, doi: 10.3390/math12070947.
- [19] F. Chen, "Research on the Performance of Network Propagation by Using the Machine Learning and Internet-of-Things Technology Integrating Model," *Computational Intelligence and Neuroscience*, vol. 2022, pp. 1-13, 2022, doi: 10.1155/2022/5480015.
- [20] K. Kolar, D. Dondorp, J. C. Zwiggelaar, J. Høyer, and M. Chatzigeorgiou, "Mesmerize Is a Dynamically Adaptable User-Friendly Analysis Platform for 2D and 3D Calcium Imaging Data," *Nature Communications*, vol. 12, no. 1, 2021, doi: 10.1038/s41467-021-26550-y.
- [21] A. Blanca, Z. Chen, D. Štefankovič, and E. Vigoda, "The Swendsen–Wang Dynamics on Trees," *Random Structures and Algorithms*, vol. 62, no. 4, pp. 791-831, 2022, doi: 10.1002/rsa.21121.
- [22] W. Feng, H. Guo, and J. Wang, "Swendsen-Wang Dynamics for the Ferromagnetic Ising Model With External Fields," 2022, doi: 10.48550/arxiv.2205.01985.
- [23] G. Aletti and I. Crimaldi, "The Rescaled Pólya Urn: Local Reinforcement and Chi-Squared Goodness-of-Fit Test," *Advances in Applied Probability*, vol. 54, no. 3, pp. 849-879, 2022, doi: 10.1017/apr.2021.56.
- [24] R. Qiao, F. Dong, X. Xie, and R. Ji, "Regional Differences, Dynamic Evolution, and Spatial Spillover Effects of Carbon Emission Intensity in Urban Agglomerations," *Environmental Science and Pollution Research*, vol. 30, no. 58, pp. 121993-122010, 2023, doi: 10.1007/s11356-023-30807-y.
- [25] X. Zheng and W. Liao, "Coupling Coordination Between Agricultural Carbon Emission Efficiency and Food Security in China: The Spatial-Temporal Evolution and Prediction," *Plos One*, vol. 20, no. 7, p. e0325026, 2025, doi: 10.1371/journal.pone.0325026.
- [26] Y. Li, C. Wen, S. Yang, M. Ma, J. Cheng, and W. Li, "Measuring High-Speed Train Delay Severity: Static and Dynamic Analysis," *Plos One*, vol. 19, no. 4, p. e0301762, 2024, doi: 10.1371/journal.pone.0301762.
- [27] J. Lara, M. Samper, and D. G. Colomé, "Load Profile Assignment for Planning and Operation Support in Distribution Networks Under Partial Smart Meter Penetration," *Processes*, vol. 14, no. 10, p. 1505, 2026, doi: 10.3390/pr14101505.
- [28] S. Kwon, J. Kim, and Z. Lee, "Advances in Search Strategy Using the Set of Brand Considerations in the Web Ecosystem," *Applied Sciences*, vol. 11, no. 8, p. 3514, 2021, doi: 10.3390/app11083514.
- [29] R. Favela, "Bridging Realities: Blockchain Adoption Dynamics and a Quantum Wormhole Framework for Spacetime-Optimized Supply Chains," 2025, doi: 10.21203/rs.3.rs-7216400/v1.
- [30] V. Krishnamurthy, R. Luo, and B. Nettasinghe, "Segregation in Social Networks: MARKOV Bridge Models and Estimation," 2021, doi: 10.1109/icassp39728.2021.9413690.
- [31] J. Culp, D. M. Ashby, A. G. George, G. C. Teskey, W. Nicola, and A. McGirr, "A Markovian Neural Barcode Representing Mesoscale Cortical Spatiotemporal Dynamics," 2024, doi: 10.1101/2024.06.29.601346.
- [32] F.-C. Jeng and Y.-S. Jeng, "Implementation of Machine Learning on Human Frequency-Following Responses: A Tutorial," *Seminars in Hearing*, vol. 43, no. 03, pp. 251-274, 2022, doi: 10.1055/s-0042-1756219.